TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 652

GOLDSTEIN'S SOLUTION OF THE PROBLEM OF THE

AIRCRAFT PROPELLER WITH A FINITE NUMBER OF BLADES

By H. B. Helmbold

Washington
December, 1931

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 652

GOLDSTEIN'S SOLUTION OF THE PROBLEM OF THE

AIRCRAFT PROPELLER WITH A FINITE NUMBER OF BLADES*

By H. B. Helmbold

The basis of the modern propeller theory was a treatise by A. Betz in 1919 on "Screw Propellers with Minimum Loss of Energy" with an addendum by L. Prandtl, which contained, for the first time, an approximate solution of the problem of the screw propeller with a finite number of blades. (Reference 1.) The Betz theory was limited to frictionless, lightly loaded propellers. Prandtl's addendum showed how the results can be extended to moderately loaded propellers. The essential points of the Betz theory are:

I.   "The flow behind a propeller with minimum loss of energy is as if the path traversed by each propeller blade (helical surface) were congealed and driven astern with a definite velocity."

II. "For symmetrical propellers the interference velocity at the blade is half as great as that at the corresponding point of the helical surface far behind the propeller."

If the number of blades is infinite, the helical surfaces stand so close together that, within the slipstream at a finite distance from the margin of the helical surfaces, there is no radial velocity, and the interference flows are everywhere perpendicular to the helical surfaces. The flow is therefore perfect within the slipstream. If w denotes the backward velocity of the helical surfaces and β their pitch angle at a distance r from the axis, then the interference velocity $w_n = w \cos \beta$, its axial component $w_a = w \cos^2 \beta$ and its tangential component $w_t = w \cos \beta \sin \beta$. (The relation between the flight speed v, the angular velocity ω, the radius r and the pitch

angle β is represented by $\tan \beta = v/\omega r$.) The circulation is represented by the path integral $\oint v_s \, ds$. The total circulation about all the z blade sections on the circumference $2\pi r$ is accordingly $z\Gamma = 2\pi r w_t$, since the tangential velocity behind the propeller is uniformly equal to w_t along the whole circumference, and the flow in front of the propeller can yet have no helix angle. According to the theorem of Kutta and Joukowsky (The result can also be obtained by the use of momentum and energy laws), the proportion of the thrust falling on the annulus of the propeller-disk area between r and $r + dr$ is $dS = \rho \omega r \, z\Gamma \, dr$ and the corresponding share of the moment is $dM = r \rho v \, z\Gamma \, dr$. Hitherto a low flow velocity as compared with the absolute velocity of the propeller blades or, in other words, a low degree of loading was tacitly assumed. When this assumption no longer applies, the derived formulas are valid.

$$\tan \varphi = \frac{v + \dfrac{w_a}{2}}{\omega r - \dfrac{w_t}{2}} = \frac{v + \dfrac{w}{2}}{\omega \, r}$$

$$w_n = w \cos \varphi; \quad w_a = w \cos^2 \varphi; \quad w_t = w \cos \varphi \sin \varphi$$

$$dS = \rho \left(\omega r - \frac{w_t}{2} \right) z\Gamma \, dr; \quad dM = r \rho \left(v + \frac{w_a}{2} \right) z\Gamma \, dr.$$

What is changed in the transition to a finite number of blades, if the quantities v, ω and w remain unchanged? Even then the interference flows are perpendicular to the helical surfaces, if the newly developed radial velocities are at first disregarded, so that the last formulas are not changed. Due to the finite distance between the helical surfaces, a circulation about their edges occurs, and radial velocities are developed in the slipstream. The flow is then no longer perfect, and the fluid avoids the helical surfaces. If the course of the tangential velocities is now followed on a circumference $2\pi r$ behind the propeller, it is found that, on the helical surfaces themselves, the same values of w_t exist as for an infinite number of blades, but that changes have taken place between them, so that the tangential velocities are smaller near the circumference and greater near the center. If the whole circulation is then considered as a path integral of the velocity at the circumference $2\pi r$, we have

$$z\Gamma = 2\pi r\, \overline{w}_t = 2\pi r\, \kappa\, w_t,$$

where $\overline{w}_t = \kappa\, w_t$ is the mean value of the tangential velocity on the circumference and κ is the corresponding mean-value factor with respect to the value w_t on the helical surfaces. According to the above, this is smaller than unity near the blade tips and greater than unity near the axis of rotation. This behavior of the mean-value factor or of the tangential velocity can be explained as follows. With an infinite number of blades the circulation increases toward the tips and then falls abruptly to zero at the tips. This is not possible with a finite number of blades, because the pressure difference between the pressure and suction side can then be eliminated around the blade tips, with a more gradual fall of the circulation to zero at the tips. With an infinite number of blades, the fluid behaves, near the axis, like a rotating solid body, due to the perfect flow but, with a finite number of blades, it has more freedom of motion between the helical surfaces and leads somewhat in the angle between two successive helical surfaces. This is due to the fact that, in this angle, the suction side of the leading propeller blade is followed by the pressure side of the next blade, and the pressure drop between the successive propeller blades forces the fluid again in the direction of rotation.*

The expressions for thrust and torque can now be written:

$$dS = \rho\left(\omega r - \frac{w_t}{2}\right)\kappa\, w_t\, 2\pi r\, dr$$

$$dM = r\,\rho\left(v + \frac{w_a}{2}\right)\kappa\, w_t\, 2\pi r\, dr.$$

---

*Another consideration yields a somewhat stronger argument. The flow between the helical surfaces must be free from rotation, so that, for an observer carried along with the helical surfaces, any fluid particle in the angle between two successive helical surfaces must have an opposite rotation. From this there follows, for the vicinity of the axis, the streamline form shown in Figure 1, which represents, in the direction of the circumference, an advance of the fluid with respect to the helical surfaces.

The task is now to determine the mean-value factor κ. The Prandtl considerations are limited to the study of the κ near the blade tips. The approximate solution is all the better, the closer the helical surfaces succeed one another, i.e., the smaller the ratio of their perpendicular distance on the margin to the circumference

$$\frac{H \cos \beta_R}{z \; 2 \pi R} = \frac{\lambda}{z \sqrt{1 + \lambda^2}}$$

where $\lambda = \tan \beta_R = \dfrac{v}{\omega R}$ the pitch angle of the propeller. For a more heavily loaded propeller we write

$$\frac{H \cos \varphi_R}{z \; 2 \pi R} = \frac{h}{z \sqrt{1 + h^2}}$$

with the pitch angle

$$h = \tan \varphi_R = \frac{v + \dfrac{w}{2}}{\omega R}$$

Prandtl's formula now reads:

$$\kappa = \frac{2}{\pi} \arccos e^{-\chi \left(1 - \frac{r}{R}\right)} \quad \text{with} \quad \chi = \frac{z}{2} \frac{\sqrt{1 + h^2}}{h}$$

In Figure 2, κ is plotted against $\chi \left(1 - \dfrac{r}{R}\right)$. Th. Troller found that this formula generally yielded a close approximation, even in cases where the conditions of theorem I were no longer fulfilled, i.e., where the velocity w is no longer independent of the radius r (reference 3), thus demonstrating the feasibility of the formula $\overline{w}_t = \kappa \, w_t$.

At the suggestion of Professor Betz, the problem was recently attacked by the Englishman, S. Goldstein and solved in exact form for the important case of a friction-less lightly loaded propeller with minimum loss of energy. (Reference 4.) The result is likewise plotted in Figure 2. Instead of Prandtl's κ curve, there is a set of curves with h and z as parameters. However, as shown by the comparison of the curves for $h = 0.1$, $z = 2$, and $h = 0.2$, $z = 4$, the course of κ probably depends chiefly on

a parameter h/z or $1/X$. The Prandtl solution, as was to be foreseen, represents the limiting value of the Goldstein solution for $h \to 0$ or $z \to \infty$.

Aside from the mean-value factor κ, the ratio of the thrusts with an infinite and with a finite number of blades $\zeta = S_\infty/S_z$ is important for practical application. In comparing the thrusts, the quantities v, ω, w, and R are to be regarded as given. Theorem III then applies: "A screw propeller with a finite number of blades z and with the thrust S_z is, for a pitch angle λ, approximately equivalent (with respect to the propeller slip $\vartheta = w/v$) to a propeller with an infinite number of blades, but somewhat greater thrust $S_\infty = \zeta S_z$." According to the above considerations, the ratio of the thrusts is calculated simply from

$$\zeta = \frac{S_\infty}{\int_0^R \kappa \, dS}$$

The conditions of theorem I yield the function ζ of $1/X$ represented in Figure 3. This graph was chosen, in order to bring the results for $z = 2$ and $z = 4$ into conformity as nearly as possible. For small values of $1/X$, the function can be approximately represented by

$$\zeta = 1 + \frac{2}{X} \approx 1 + 4\frac{h}{z}.$$

Knowledge of the propeller slip $\vartheta = \frac{w}{v}$ is necessary for calculating the propeller shape. For moderately loaded propellers of the Betz type (reference 2), we have

$$\vartheta = \frac{\varphi}{\varphi'}\left(-1 + \sqrt{1 + \frac{\varphi'}{\varphi^2}\,\zeta\,c_s}\right)$$

and approximately

$$\vartheta = \frac{\varphi}{\varphi + \varphi'}\left(-1 + \sqrt{1 + \frac{\varphi + \varphi'}{\varphi^2}\,\frac{\zeta\,c_l}{1 + \frac{2}{3}\frac{\zeta}{\lambda}}}\right).$$

Given are the pitch angle λ, the loading $c_s = \dfrac{S}{\dfrac{\rho}{2}\,v^2\,\pi\,R^2}$

or, instead of this, the performance rating $c_l = \dfrac{N}{\dfrac{\rho}{2} v^3 \pi R^2}$

and the fineness ratio ϵ. The quantities φ and φ' are functions of the pitch angle h.

$$\varphi = 1 - h^2 \ln\left(1 + \frac{1}{h^2}\right); \quad \varphi' = 2\varphi - \frac{1}{1 + h^2},$$

represented in Figure 4. Since the pitch angle is not known at first, for the calculation of φ, φ' and ζ, h is first replaced by λ, the pitch angle

$$h = \lambda\left(1 + \frac{\vartheta}{2}\right)$$

is calculated with the found ϑ, and with this an improved value of ϑ is found. The method converges quite rapidly. The blade chord is found from

$$\frac{t_a}{R} = \frac{4\pi r \kappa w_t \cos\varphi}{z\left(\omega r - \frac{w_t}{2}\right)} =$$

$$= 4\pi \frac{\vartheta \lambda h}{z} \frac{\kappa \frac{R}{r}}{\left[1 + \lambda h \left(\frac{R}{r}\right)^2\right] \sqrt{1 + \left(h \frac{R}{r}\right)^2}}$$

where $t_a = c_a t$, the "lift chord," i.e., the blade chord for the lift coefficient $c_a = 1$. Figure 5 shows several examples of distribution of the blade chord along the radius for the case of a lightly loaded propeller $(h \rightarrow \lambda)$. For better comparison, $c_a = 1$ was put for two-blade propellers, $c_a = 0.5$ for four-blade propellers and $\vartheta = 1$ throughout. For the sake of completeness we will also include the approximation formula for the propeller efficiency. (Reference 2.)

$$\eta = \frac{1}{1 + \frac{\vartheta}{2}} \frac{1 - 2\epsilon h}{1 + \frac{2\epsilon}{3h}}$$

References

1. Betz, A.: Screw Propellers with Minimum Loss of En-
       ergy. With an addendum by L. Prandtl. Vier Ab-
       handlungen zur Hydrodynamik und Aerodynamik.
       Gottingen, 1927.

2. Helmbold, H. B.: The Aerodynamics of the Propeller.
       Zeitschrift für Flugtechnik und Motorluftschif-
       fahrt, July and August, 1924, pp. 150, 170.

3. Troller, Th.: On the Vortex Theory of Propellers.
       Zeitschrift für angewandte M. & M., Dec., 1928,
       pp. 426-30.

4. Goldstein, S.: On the Vortex Theory of Screw Propel-
       lers. Proc. Roy. Soc. of London, A., Vol. 123,
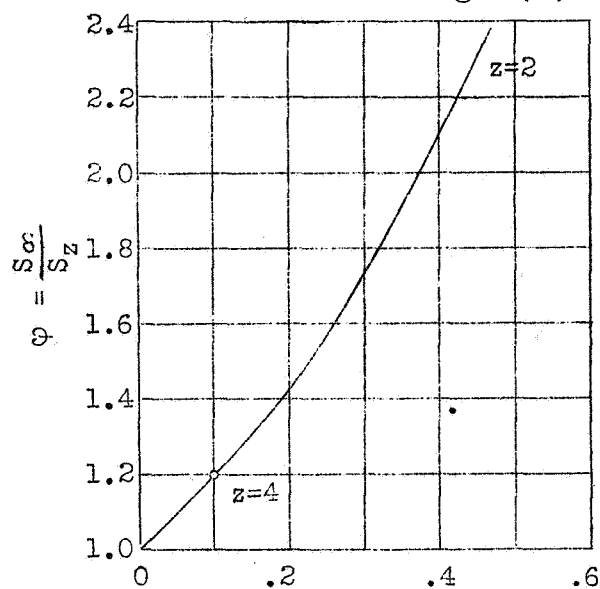   1929, pp. 440-65.

Fig.1   Flow near pro-
        peller axis.

$$\frac{1}{\chi} = \frac{2}{z}\frac{h}{\sqrt{1+h^2}}$$

Fig.3 Ratio of
       the thrusts
with infinite and
with finite number
of blades plotted
against the ratio
of the perpendicular
distance of the
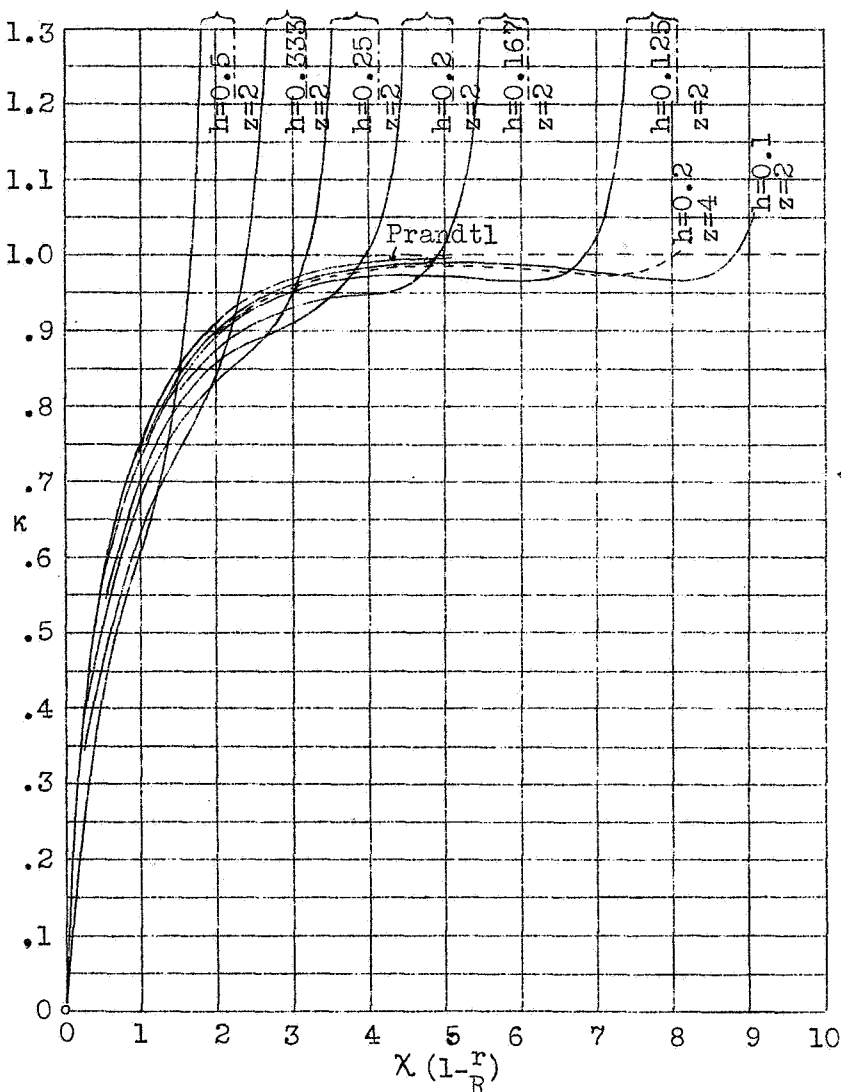successive helical
surfaces to the
propeller-disk
circumference.

Fig.2 Circulation
       ratio with
finite and with
infinite number of
blades $\kappa = \dfrac{zI}{2\pi r\, w_t}$
(mean-value factor)
plotted against the
relative distance
R-r/R from the blade
tip and against the
ratio of the propel-
ler-disk circumference
$2\pi R$ to the perpendic-
ular distance of the
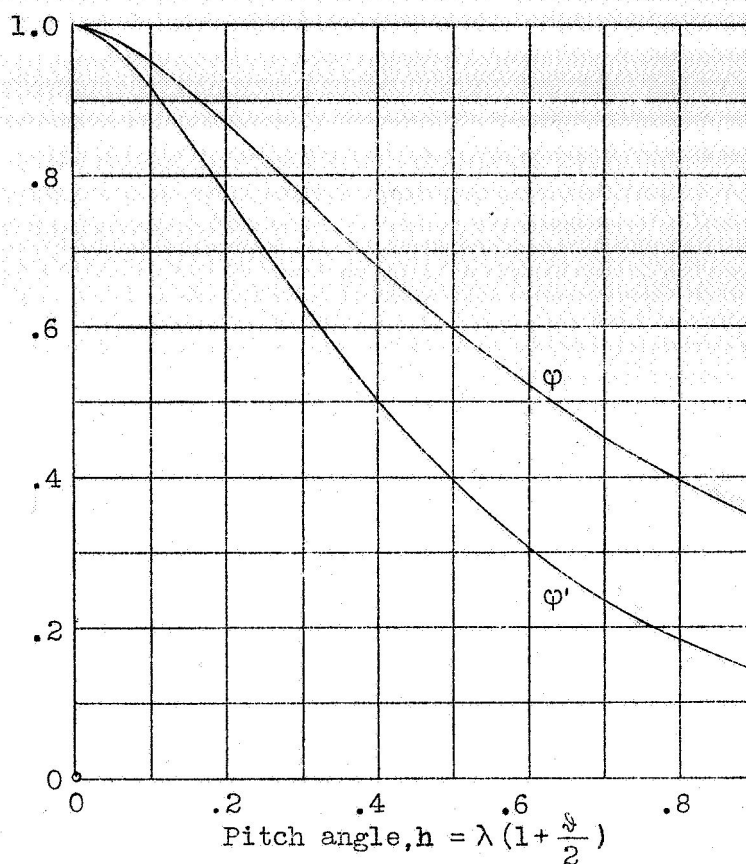successive helical
surfaces at the blade
tips.

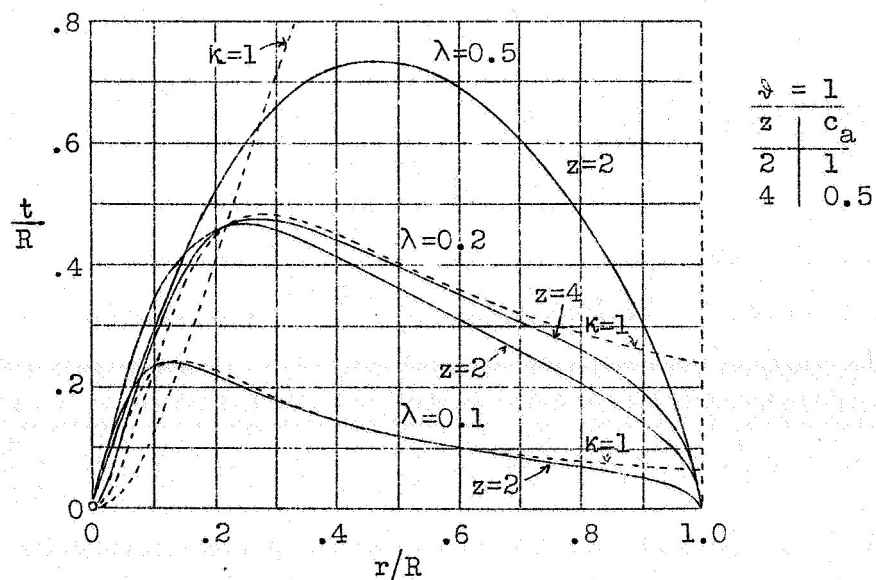Fig.4 Auxiliary diagram for computing the propeller slip.

Fig.5 Examples of blade-chord distribution along radius for various blade chords and pitch angles.